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INFINITARY MODEL THEORY OF ABELIAN GROUPS

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ABSTRACT
The paper is a survey of results in the model theory of abelian groups, dealing with two sorts of problems: finding invariants which classify groups up to $L_{\lambda\kappa}$-equivalence; and determining whether certain classes of groups are definable in $L_{\lambda\kappa}$.

Our aim in this paper is to survey the model theory of abelian groups with emphasis on infinitary model theory and on the interesting role which set theory has come to play in this subject. Rather than give a comprehensive account of results, we shall present a sampling which illustrates the types of results known and the methods used in their proofs. We shall deal with two types of problems, as follows. Let $\mathscr{A}$ be a class of abelian groups. The *definability problem* asks: is $\mathscr{A}$ definable in $L_{\lambda\kappa}$, where λ and κ are cardinals or ∞? If $\mathscr{A}$ is definable in $L_{\lambda\kappa}$, the *classification problems* asks for a characterization of the $L_{\lambda\kappa}$-equivalence classes of $\mathscr{A}$.

Throughout this paper "group" will mean abelian group; $\mathbf{Q}$ denotes the additive group of the rationals; and $\mathbf{Z}$ denotes the additive group of the integers. Throughout, κ and λ will be used to denote infinite cardinals; and $|A|$ denotes the cardinality of A. Also A^{κ} (respectively $A^{(\kappa)}$) denotes the direct product (respectively, direct sum) of κ copies of the group A. The notation $A \equiv_{\infty\kappa} A'$ means that the groups A and A' are $L_{\infty\kappa}$-equivalent, i.e., they satisfy the same sentences of $L_{\infty\kappa}$.

1. Classification

The work of W. Szmielew [17] provides a complete solution to the classification problem for finitary logic. That is, Szmielew gave a complete set of

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invariants for the relation of elementary equivalence of groups. To save space we shall not state her result in full, but in order to indicate its character we shall describe the Szmielew invariants for torsion groups.

Given a group A and a prime p we define $p^\nu A$ for any ordinal ν by induction on ν. Let $p^0A = A$; $p^{\nu+1}A = \{pa : a \in p^\nu A\}$; and if σ is a limit ordinal, $p^\sigma A = \bigcap_{\nu<\sigma} p^\nu A$. Also define $p^\nu A[p] = \{a \in p^\nu A : pa = 0\}$. For any $t = (p, \nu)$, let $A_t = p^\nu A[p]/p^{\nu+1}A[p]$. For any group B and any cardinal κ, let $\mathrm{rk}_\kappa(B)$ denote the minimum of κ and the rank of B. (If $pB = 0$—e.g. if $B = A_t$—then B is naturally a vector space over the field of order p and the rank of B is the same as the dimension of B as a vector space.)

THEOREM 1.1 (Szmielew). *If A and A' are torsion groups, then A is elementarily equivalent to B if and only if*:

i) $\mathrm{rk}_\omega(A_t) = \mathrm{rk}_\omega(A'_t)$ *for all* $t = (p, n)$, $n \in \omega$;

ii) $\mathrm{rk}_\omega p^n A[p] = \mathrm{rk}_\omega p^n A'[p]$ *for all* $n \in \omega$.

(See Eklof–Fisher [8] for a different proof of Szmielew's results as well as other results in the finitary model theory of groups.)

For infinitary languages, results are more fragmentary. Let $\mathcal{T}$ be the class of torsion groups, i.e. the class of groups which are models of the sentence

$$\forall x \bigvee_{n\neq 0} nx = 0.$$

Since $\mathcal{T}$ is definable in $L_{\omega_1\omega}$ it is natural to ask for its equivalence classes with respect to $L_{\omega_1\omega}$ or, more generally, $L_{\lambda\omega}$. The answer is provided by the following result.

THEOREM 1.2 (Barwise–Eklof [2]). *Let λ be a regular cardinal, and let $A, A' \in \mathcal{T}$. Then $A \equiv_{\lambda\omega} A'$ if and only if*

i) $\mathrm{rk}_\omega(A_t) = \mathrm{rk}_\omega(A'_t)$ *for all* $t = (p, \nu)$, $\nu < \lambda$;

ii) $\mathrm{rk}_\omega(p^\nu A[p]) = \mathrm{rk}_\omega(p^\nu A'[p])$ *for all p and all* $\nu < \lambda$.

The proof uses the back-and-forth criterion for $L_{\lambda\omega}$-equivalence and an extension of the methods used to prove Ulm's Theorem. (See Barwise [1] for an exposition of the proof.) In fact Ulm's Theorem can be recovered from the statement of Theorem 1.2 (with $\lambda = \omega_1$) by applying Scott's Theorem:

ULM'S THEOREM 1.3. *If A and A' are* countable *torsion groups then $A \cong A'$ if and only if*

i) $\mathrm{rk}(A_t) = \mathrm{rk}(A'_t)$ *for all* $t = (p, \nu)$, $\nu < \omega_1$;

ii) $\mathrm{rk}\, p^\nu A[p] = \mathrm{rk}\, p^\nu A'[p]$ *for all p and all* $\nu < \omega_1$.

Any group A can be written as $A = A_r \oplus A_d$, where A_d is divisible and A_r has no non-zero divisible subgroups. Since divisible groups are direct sums of copies of $\mathbf{Q}$ and $Z(p^\infty)$, the p-primary part of $\mathbf{Q}/\mathbf{Z}$, their structure is transparent. For *reduced* groups, i.e. groups A such that $A_d = \{0\}$, part (ii) of Theorems 1.2 and 1.3 is unneeded. Also if the *p-length of A* (i.e. the least μ such that $p^\mu A = p^{\mu+1}A$, or the least μ such that $p^\mu A = A_d$) is $\geqq \lambda$ and the same is true for A', then (ii) is unnecessary; indeed in that case $\mathrm{rk}_\kappa(p^\nu A) = \kappa = \mathrm{rk}_\kappa(p^\nu A)$ for all $\nu < \lambda$.

Theorem 1.2 can be generalized to mixed groups of torsion-free rank one using results of Megibben [13] on extending Ulm's Theorem.

Before giving a classification theorem for a class defined in $L_{\lambda\kappa}$ where $\kappa \geqq \omega_1$ we need to make some group-theoretic definitions. A group A is called *completely decomposable* if it is a direct sum of rank one groups (see Fuchs [10], Ch. XIII). There is a complete set of invariants for completely decomposable groups. In order to describe the invariants, let us first consider the rank one groups. These are either torsion or torsion-free. The torsion rank one groups are simply the cyclic groups, $Z(p^n)$, of prime power order, or the divisible groups $Z(p^\infty)$.

If A is any group and a is a non-zero element of A, let $\chi(a)$, the *characteristic* of a, be the sequence $\langle k_1, k_2, \cdots \rangle$ where k_i is the largest $n \in \omega$ such that p_i^n divides a in A, if it exists, or $k_i = \infty$ otherwise. (Here p_i is the i-th prime.) If A is torsion-free, let $\langle a \rangle_*$, *the pure closure of* $\langle a \rangle$ *in* A, be the subgroup $\{\frac{m}{n} a \in A \mid n$ divides a in $A\}$. If A has rank one then obviously $\langle a \rangle_* = A$, A is isomorphic to a subgroup of $\mathbf{Q}$, and $\chi(a)$ determines A up to isomorphism. However A does not determine $\chi(a)$ uniquely. For example, if $A = \mathbf{Z}$, $\chi(1) = (0, 0, \cdots 0, \cdots)$, $\chi(2) = (1, 0, \cdots 0)$, $\chi(84) = (2, 1, 0, 1, 0, \cdots 0, \cdots)$, etc. We can define an equivalence relation on characteristics as follows. If $\chi = \langle k_1, k_2, \cdots \rangle$ and $\chi' = \langle k'_1, k'_2, \cdots \rangle$ then χ is *equivalent* to χ' if and only if $k_i = k'_i$ for all but finitely many i and if $k_i \neq k'_i$ then k_i and k'_i are both finite. An equivalence class of characteristics will be called a (torsion-free) *type*. It is not hard to see that there is a one-one correspondence between torsion-free rank one groups and torsion-free types. Where convenient we shall confuse characteristics with their equivalence classes.

The relation $\chi \geqq \chi'$ if and only if $k_i \geqq k'_i$ for all i induces a partial ordering on the set of types. If A is any group and t is any torsion-free type, define $A(t) = \{a \in A \mid a$ is torsion-free and $\chi(a) \geqq t\}$ and $A^*(t) = \{a \in A \mid a$ is torsion-free and $\chi(a) > t\}$. Then these are subgroups of A—we include 0 by convention—and we define $A_t = A(t)/A^*(t)$.

THEOREM 1.4. (Baer). *If A is a completely decomposable group—say $A = \bigoplus_i R_i$ where R_i is rank one of type t_i—then $A_t \cong \bigoplus\{R_i \mid t_i = t\}$. Hence if A and A' are completely decomposable, $A \cong A'$ if and only if $\mathrm{rk}(A_t) = \mathrm{rk}(A'_t)$ for all torsion-free types t.*

A group A is called *κ-separable* if every subset of A of cardinality $< \kappa$ is contained in a completely decomposable direct summand of A of rank $< \kappa$. The problem with this notion from a logical point of view is that it is not obvious that the class, $\mathcal{S}_\kappa$, of κ-separable groups forms an $L_{\infty\kappa}$-elementary class. (We shall consider this question in section 2). So let us define a group A to be *weakly κ-separable* if every subset of A of cardinality $< \kappa$ is contained in a completely decomposable subgroup B of rank $< \kappa$ such that B is *κ-pure in A*, i.e. B is a direct summand of C whenever $B \subseteq C \subseteq A$ and C/B has rank $< \kappa$. Then the class, $\mathcal{W}_\kappa$, of weakly κ-separable groups is easily seen to be definable in $L_{\kappa^+\kappa}$.

Let $\mathfrak{T} = \{t \mid t = (p, n),\ p \text{ prime},\ n \in \omega\} \cup \{t \mid t \text{ is a torsion-free type}\}$. The following generalizes results in Eklof [5]. The restriction to uncountable κ is only for convenience.

THEOREM 1.5. *Let $\kappa \geqq \omega_1$. If $A, A' \in \mathcal{W}_\kappa$ then the following are equivalent:*

1) $A \equiv_{\infty\kappa} A'$;
2) $A \equiv_{\kappa\kappa} A'$;
3) i) $\mathrm{rk}_\kappa(A_t) = \mathrm{rk}_\kappa(A'_t)$ *for all* $t \in \mathfrak{T}$,
   ii) $\mathrm{rk}_\kappa p^\omega A[p] = \mathrm{rk}_\kappa p^\omega A'[p]$ *for all primes* p.

PROOF. (1) $\Rightarrow$ (2) is obvious; (2) $\Rightarrow$ (3) is true because the invariants can be described in the language $L_{\kappa\kappa}$. We prove (3) $\Rightarrow$ (1) by defining a non-empty set $\mathcal{I}$ of "partial isomorphisms" from A to A' (i.e. isomorphisms $f: B \to B'$ where B (respectively B') is a subgroup of A (respectively A')) such that for any $f \in \mathcal{I}$ and any subset X of A (respectively A') of cardinality $< \kappa$ there exists $\tilde{f} \in \mathcal{I}$ such that $f \subseteq \tilde{f}$ and $X \subseteq \mathrm{dom}(\tilde{f})$ (respectively $X \subseteq \mathrm{rge}(\tilde{f})$) (cf. Calais [4] or Benda [3]). We may assume that A and A' are reduced since the invariants $\mathrm{rk}_\kappa p^\omega A[p]$ and $\mathrm{rk}_\kappa(A_{t_0})$ (where $t_0 =$ type of $\mathbf{Q}$) determine the structure of the divisible part of A (which is a definable subgroup since $\kappa \geqq \omega_1$: cf. Theorem 2.2). Hence we are interested only in the invariants $\mathrm{rk}_\kappa(A_t)$ where $t \in \mathfrak{T}' = \mathfrak{T} - \{t_0\}$. Let $\mathfrak{E} = \{t \in \mathfrak{T}' : \mathrm{rk}_\kappa(A_t) < \kappa\}$ and for $t \in \mathfrak{E}$, let $\lambda_t = \mathrm{rk}(A_t) = \mathrm{rk}(A'_t)$. Let $\mathcal{I}$ be the set of all partial isomorphisms $f: B \to B'$ such that B is completely decomposable, $|B| < \kappa$, and B (respectively B') is κ-pure in A (respectively A') and such that

$$(*)\quad \text{for all } t \in \mathfrak{E},\ \text{ if } \operatorname{rk}(B_t) = \lambda_t \ \text{ then } (B(t) + A^*(t))/A^*(t) = A_t$$
$$\text{and } (B'(t) + A'^*(t))/A'^*(t) = A'_t.$$

(If $t = (p, n)$ let $A(t) = p^n A[p]$ and $A^*(t) = p^{n+1}A[p]$ so that $A_t = A(t)/A^*(t)$, as in the torsion-free case.) Condition $(*)$ implies that there is no pure subgroup of A (respectively A') of the form $B \oplus R$ (respectively $B' \oplus R$) where R is rank one of type t.

Let $f \in \mathscr{I}$. The symmetry of the situation means that it is sufficient to prove that if X is a subset of A of cardinality $< \kappa$, then there exists $\tilde{f} \in \mathscr{I}$ with $f \subseteq \tilde{f}$ and $X \subseteq \operatorname{dom}(\tilde{f})$. Since A is weakly κ-separable there exists a completely decomposable κ-pure subgroup C_0 of A of cardinality $< \kappa$ containing $B \cup X$. Let $\mathfrak{E}_0 = \{t \in \mathfrak{E} : \operatorname{rk}(C_t) = \lambda_t\}$. For each $t \in \mathfrak{E}_0$, choose a subset Z_t of $A(t)$ of cardinality λ_t which is a set of representatives for the elements of $A_t = A(t)/A^*(t)$. Note that

$$\sum_{t \in \mathfrak{E}_0} \lambda_t < \kappa$$

since $|C_0| < \kappa$. Hence there exists a completely decomposable κ-pure C_1 of cardinality $< \kappa$ and containing $C_0 \cup \bigcup \{Z_t : t \in \mathfrak{E}_0\}$. Say $C_1 = C_0 \oplus \bigoplus_{i \in I} R_i$ where each R_i is rank one (torsion-free) of type $t_i \in \mathfrak{T}'$. (Here we use the fact that C_0 is κ-pure in A and that a direct summand of a completely decomposable group is completely decomposable; see Fuchs [9, theor. 18.1], and Fuchs [10, theor. 86.7].) Let $\tilde{I} = \{i \in I \mid t_i \in \mathfrak{E}_0\}$ and let $\tilde{C} = C_0 \oplus \bigoplus_{i \in \tilde{I}} R_i$ and write $C_1 = \tilde{C} \oplus D$. Now for $t \in \mathfrak{E}_0$, $(C_1(t) + A^*(t))/A^*(t) = A_t$ by construction, and hence $(\tilde{C}(t) + A^*(t))/A^*(t) = A_t$ since $D_t = 0$. Thus $\tilde{C}$ satisfies $(*)$.

Since B is κ-pure in A, $\tilde{C} = B \oplus \hat{B}$ where, say, $\hat{B} = \bigoplus_{t \in \mathfrak{T}'} R(t)^{(\sigma_t)}$, where $R(t)$ is the rank one group of type t. To prove that f extends to a function $\tilde{f} \in \mathscr{I}$ with $X \subseteq \operatorname{dom}(\tilde{f})$ it suffices to prove that there is an extension $\tilde{C}' = B' \oplus \hat{B}'$ of B' which is completely decomposable and κ-pure in A' and satisfies $(*)$ such that $\hat{B} \cong \hat{B}'$. But this follows—by means of a construction like that above—from the hypothesis that $\operatorname{rk}_\kappa(A_t) = \operatorname{rk}_\kappa(A'_t)$ and from the condition $(*)$ which implies that either $\operatorname{rk}_\kappa(B_t) < \operatorname{rk}_\kappa(A_t)$ or $\sigma_t = 0$. The latter ensures that if we need copies of $R(t)$ in $\hat{B}'$, then we can certainly produce them since $\operatorname{rk}_\kappa(B'_t) < \operatorname{rk}_\kappa(A'_t)$. ■

COROLLARY 1.6. *Every weakly κ-separable group is $L_{\infty\kappa}$-equivalent to a completely decomposable group.*

PROOF. Let $A \in \mathscr{W}_\kappa$. We may assume A is reduced. Let $A' = \bigoplus_{t \in \mathfrak{T}'} R_t^{(\lambda_t)}$ where R_t is rank one of type t and $\lambda_t = \operatorname{rk}_\kappa(A_t)$. Then $A \equiv_{\infty\kappa} A'$. ■

The following gives some information on the existence of groups which are $L_{\infty\kappa}$-equivalent to completely decomposable groups but which are not completely decomposable. The proof is based on a generalization of the construction given in Eklof [6].

THEOREM 1.7. *Let $\kappa = \aleph_n$ for any $n < \omega$. If f is a function from $\mathfrak{T}'$ to the set of cardinals $\leqq \kappa$ satisfying either*

i) *$f(t) = \kappa$ for some torsion-free $t \in \mathfrak{T}'$; or*

ii) *for some p, $f((p, n)) = \kappa$ for arbitrarily large n,*

then there exists $A \in \mathcal{W}_\kappa$ of cardinality κ such that A is not completely decomposable and $\mathrm{rk}_\kappa(A_t) = f(t)$ for all $t \in \mathfrak{T}'$.

Using an idea due to Gregory [11] (see Eklof [6] for more details) we can generalize Theorem 1.7 under an additional set-theoretic hypothesis.

THEOREM 1.8 ($V = L$). *If κ is regular and not weakly compact then Theorem 1.7 is true for κ.*

On the other hand, Shelah has proved the following results. (For (i) see Shelah [16].)

THEOREM 1.9 (Shelah).

i) *If κ is singular or weakly-compact and A is weakly κ-separable and of cardinality κ then A is completely-decomposable.*

ii) *If it is consistent with ZFC that there exists a supercompact cardinal, then it is consistent with ZFC that every weakly-$2^{\aleph_0}$-separable group is completely decomposable.*

It is open whether for $\kappa = \aleph_{\omega+1}$ it is provable in ZFC that there exists a weakly κ-separable group of cardinality κ which is $L_{\infty\kappa}$-equivalent to a completely decomposable group but is not completely decomposable.

It would also be interesting to obtain classification theorems for $L_{\infty\kappa}$-equivalence (where $\kappa \geqq \omega_1$) for classes of non-separable groups which arise in nature, for example, for $\mathcal{T}$, the class of torsion groups or for natural subclasses of $\mathcal{T}$.

2. Definability

We shall consider the definability problem for some of the classes considered in section 1. First, let $\mathcal{R}$ be the class of reduced groups. The following is proved in Barwise–Eklof [2] as a consequence of Theorem 1.2.

THEOREM 2.1. *$\mathcal{R} \cap \mathcal{T}$ (i.e. the class of reduced torsion groups) is not definable in $L_{\infty\omega}$.*

On the other hand we have the following:

THEOREM 2.2.

i) *$\mathcal{R}$ is closed under $L_{\infty\omega}$-equivalence, i.e. if $A \equiv_{\infty\omega} B$ then $A \in \mathcal{R}$ if and only if $B \in \mathcal{R}$.*

ii) *The class of torsion-free reduced groups is definable by a sentence of $L_{\omega_1\omega}$.*

iii) *$\mathcal{R}$ is definable by a sentence of $L_{\omega_1\omega_1}$.*

PROOF.

i) Suppose $A \equiv_{\infty\omega} B$. If A is not reduced then A contains a countable non-zero divisible subgroup A'. Since $A \equiv_{\infty\omega} B$ there exists a non-empty set $\mathcal{I}$ of partial isomorphisms from A to B such that for every $f \in \mathcal{I}$ and every $a \in A$ (respectively $b \in B$) there exists $\tilde{f} \in \mathcal{I}$ with $f \subseteq \tilde{f}$ and $a \in \mathrm{dom}(\tilde{f})$ (respectively $b \in \mathrm{rge}(\tilde{f})$). By induction we can define a chain $\{f_n \mid n \in \omega\}$ of elements of $\mathcal{I}$ such that f_n contains the first n elements of A' (in some fixed well-ordering of type ω). Then $\bigcup f_n$ is an isomorphism of A' with a submodule of B. Hence B is not reduced.

ii) In a torsion-free group division is unique, and hence a torsion-free group is reduced if and only if no non-zero element is divisible by all $n \neq 0$, if and only if the group is a model of

$$\forall x \left[x \neq 0 \rightarrow \bigvee_{n \neq 0} \forall y (ny \neq x) \right].$$

iii) A group A is reduced if and only if it does not contain a copy of $\mathbf{Q}$ or $Z(p^\infty)$ for some p. Since these are countable groups, it is clear that this condition is expressible as a sentence of $L_{\omega_1\omega_1}$. ■

Our main interest is in the definability of the class, $\mathcal{S}_\kappa$, of κ-separable groups. The difficulty in expressing the property of being κ-separable is in talking about direct summands of arbitrarily large groups. We begin with the case of ω-separable groups—usually simply called *separable* groups.

THEOREM 2.3.

i) *$\mathcal{S}_\omega \cap \mathcal{T}$ (i.e. the class of separable torsion groups) is definable by a sentence of $L_{\omega_1\omega}$.*

ii) $\mathcal{S}_\omega$ is not definable in $L_{\infty\omega}$.

PROOF.

i) It suffices to consider p-primary groups since a torsion group is the direct sum of p-primary groups, where p ranges over the primes. It is a standard result of group theory that a *reduced* p-group A is separable if and only if A does not have elements of infinite height, i.e. $p^\omega A = \{0\}$. Since a divisible group is obviously separable, an arbitrary p-group is separable if and only if A satisfies $p^\omega A = p^{\omega+1} A$ (in which case $p^\omega A = A_d$), i.e. A is a model of

$$\forall x\left[\left(\bigwedge_n (\exists y(p^n y = x))\right) \rightarrow \exists z\left(\bigwedge_n (\exists y(p^n y = z)) \wedge (pz = x)\right)\right].$$

ii) There is a subgroup H of $\mathbf{Z}^\omega$ which is not separable (cf. Fuchs [10], p. 122, Ex. 3). Now $\mathbf{Z}^\omega$ is $\aleph_1$-free, i.e. every countable subgroup is free. Hence H is $\aleph_1$-free and by a result of Kueker (see Kueker [12] or Barwise [1]) H is $L_{\infty\omega}$-equivalent to a free group. Since a free group is separable, the desired result follows. ■

For uncountable κ we have the following observation.

THEOREM 2.4. *For any $\kappa \geqq \omega_1$, $\mathcal{S}_\kappa$ is definable in $L_{\infty\kappa} \Leftrightarrow \mathcal{W}_\kappa = \mathcal{S}_\kappa$.*

PROOF. The implication ($\Leftarrow$) is obvious from the remarks in section 1. For the converse, suppose $\mathcal{S}_\kappa$ is definable by the sentence θ of $L_{\infty\kappa}$. We must prove that if $A \in \mathcal{W}_\kappa$ then $A \in \mathcal{S}_\kappa$. By Corollary 1.6 there is a completely decomposable group B such that $A \equiv_{\infty\kappa} B$. But then since B obviously belongs to $\mathcal{S}_\kappa$, $B \models \theta$, so $A \models \theta$ and A belongs to $\mathcal{S}_\kappa$ also. ■

In order to obtain negative results for the case $\kappa \geqq \omega_1$ we shall assume the Axiom of Constructibility ($V = L$) and make use of Shelah's result on the Whitehead problem.

THEOREM 2.5 (Shelah [15]). ($V = L$). *If A is not free then there exists a short exact sequence $0 \rightarrow \mathbf{Z} \rightarrow A' \rightarrow A \rightarrow 0$ which does not split.*

We also make use of a result of Gregory [11]. A group A is called *κ-free* if every subgroup of A of cardinality $< \kappa$ is free.

THEOREM 2.6 (Gregory). ($V = L$). *For any regular κ which is not weakly-compact there exists $A \in \mathcal{W}_\kappa$ such that A has cardinality κ and A is κ-free but A is not free.*

LEMMA 2.7 (Mekler). *If $0 \rightarrow \mathbf{Z} \xrightarrow{\mu} A' \xrightarrow{\pi} A \rightarrow 0$ is exact and $A \in \mathcal{W}_\kappa$ and A is κ-free, then $A' \in \mathcal{W}_\kappa$ and A' is κ-free. If the sequence does not split, then $A' \notin \mathcal{S}_\omega$.*

PROOF. It suffices to prove that if B is a κ-pure, free subgroup of A then $\pi^{-1}(B)$ is a κ-pure, free subgroup of A'. Now $0 \rightarrow \mathbf{Z} \rightarrow \pi^{-1}(B) \rightarrow B \rightarrow 0$ is exact so, since B is free, the sequence splits and $\pi^{-1}(B) \cong B \oplus \mathbf{Z}$ is free. If $\pi^{-1}(B) \subseteq C \subseteq A$ and $C/\pi^{-1}(B)$ has rank $< \kappa$ then $\pi(C)/B$ has rank $< \kappa$. Since B is κ-pure in A, $\pi(C)/B$ is free so, since π induces an isomorphism of $C/\pi^{-1}(B)$ onto $\pi(C)/B$, $\pi(C)/B$ is free. Therefore A' is weakly κ-separable. If the sequence does not split, then $\mu(\mathbf{Z})$ is not a direct summand of A'. But if $A' \in \mathcal{S}_\omega$, then $\mu(\mathbf{Z})$ is contained in a direct summand B of A which is free and finitely-generated. Since $B/\mu(\mathbf{Z})$ is torsion-free, the fundamental theorem of abelian groups implies that $B/\mu(\mathbf{Z})$ is free. But then $\mu(\mathbf{Z})$ is a direct summand of B and hence of A, a contradiction. ■

THEOREM 2.8 ($V = L$). *For any infinite κ, $\mathcal{S}_\kappa$ is not definable in $L_{\infty\infty}$.*

PROOF. It suffices to show that for arbitrarily large κ there exist groups A' and A'' such that $A' \equiv_{\infty\kappa} A''$, $A'' \in \mathcal{S}_\kappa$, but $A' \notin \mathcal{S}_\omega$. (Note that if $\lambda \leqq \kappa$ then $\mathcal{S}_\kappa \subseteq \mathcal{S}_\lambda$.). Let A be as in Theorem 2.6. Then by Theorem 2.5 there is a short exact sequence $0 \to \mathbf{Z} \to A' \to A \to 0$ which does not split. By Lemma 2.7 $A' \in \mathcal{W}_\kappa$ and A' is κ-free but $A' \notin \mathcal{S}_\omega$. Let A'' be the free group of rank κ. By Theorem 1.5, $A' \equiv_{\infty\kappa} A''$. ■

On the other hand, if it is consistent with ZFC that there exists a strongly compact cardinal then it is consistent with ZFC that Theorem 2.8 is false. Before proving this we give a proof of a result of Mekler [14].

THEOREM 2.9 (Mekler). *Let κ be a strongly compact cardinal. Let B be a κ-pure subgroup of A such that B is a direct sum of groups of cardinality $< \kappa$. Then B is a direct summand of A.*

PROOF. Say $B = \bigoplus_{i \in I} B_i$ where $|B_i| < \kappa$. Let $S = \bigcup_{i \in I} B_i$. Then every element of B can be written as a linear combination of elements of S in $< \kappa$ different ways. (We consider only linear combinations of distinct elements where all the coefficients are non-zero.) Let L be the language consisting of a binary function symbol $+$, a constant symbol c_a for each element $a \in A$, and two unary predicate symbols U, V. Let T be a set of sentences of $L_{\kappa\kappa}$ consisting of: the diagram of A; a sentence which says that the universe equals $\langle U \rangle \oplus V$ ($\langle U \rangle$ is the subgroup generated by U); the sentence $U(c_b)$ for each $b \in S$; and for each $b \in B$ a sentence, θ_b, which says that if c_b is a linear combination of elements of U then these elements are in S. (The sentences θ_b are the only infinitary sentences in T; and $\theta_b \in L_{\kappa\kappa}$ because of the observation above.) Since B is κ-pure in A, every subset, T', of T of cardinality $< \kappa$ has a model. (Indeed T' has a model A' where $B \subseteq A' \subseteq A$ and $U = S$.) Therefore T has a model. Hence $A \subseteq C$ where $C = \langle U \rangle \oplus V$ and $S \subseteq U$. Now B is a direct summand of

$\langle U\rangle$; indeed, because of the sentences θ_b, $\langle U\rangle = \langle S\rangle \oplus \langle U - S\rangle = B \oplus \langle U - S\rangle$. Then $A = B \oplus (A \cap (\langle U - S\rangle \oplus V))$. ■

THEOREM 2.10. *If there exists a strongly compact cardinal, κ, then for any cardinal λ, $\mathcal{S}_\lambda$ is definable in $L_{\infty\infty}$.*

PROOF. Indeed if $\rho = \max\{\kappa, \lambda\}$ then $\mathcal{S}_\lambda$ is definable by a sentence ψ of $L_{\infty\rho}$. Let ψ be the sentence which says that every set of cardinality $< \lambda$ is contained in a completely-decomposable subgroup of cardinality $< \lambda$ which is κ-pure in the whole group. Since a completely decomposable group is a direct sum of countable groups, a completely-decomposable subgroup which is κ-pure in a group is in fact a direct summand by Theorem 2.9. ■

We do not know if—for example—it is possible to prove in ZFC that $\mathcal{S}_{\omega_1}$ is not definable in $L_{\infty\omega_1}$. To prove this it is necessary by Theorem 2.4 to show that $\mathcal{W}_{\omega_1} \neq \mathcal{S}_{\omega_1}$. The following result, which extends an observation of Mekler's about Shelah's result on the Whitehead problem, indicates that to prove $\mathcal{W}_{\omega_1} \neq \mathcal{S}_{\omega_1}$ one should look to groups of cardinality $\geqq 2^{\aleph_0}$. (See Eklof [7] for a proof; MA stands for Martin's Axiom.)

THEOREM 2.11 (MA). *If $A \in \mathcal{W}_{\omega_1}$, A is torsion-free and homogeneous (i.e. all elements of A are of the same type) and $|A| < 2^{\aleph_0}$, then $A \in \mathcal{S}_{\omega_1}$.*

By Shelah's result (Theorem 1.9 (ii)), for $\kappa = 2^{\aleph_0}$ it is consistent with ZFC that $\mathcal{S}_\kappa$ is definable in $L_{\infty\kappa}$ (assuming the consistency of the existence of a supercompact cardinal).

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